Spectral Subspaces for Compact Actions.

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Abstract: We introduce a spectral subspace theory for the action of compact groups on Banach spaces, and consider some applications of our theory, particularly for von Neumann algebras.

Introduction.

In [2, 3, 4] a spectral subspace theory has been developed for the action of locally compact abelian groups on Banach spaces. This has been a powerful technique in the study of the representations of such groups by *-automorphisms, on C^*-algebras and W^*-algebras [2, 3, 4, 7, 8, 9, 14]. In this paper we introduce a spectral subspace theory for the action of compact, not necessarily abelian groups on Banach spaces, and consider some applications, particularly to von Neumann algebras.

Some objects related to our spectral subspaces for compact actions have arisen in the work of J.E. Roberts [10, 11, 12], particularly for properly infinite von Neumann algebras. His formalism has been utilised in [1], and during the course of this paper, we shall see how our concepts and results compare with those in [1, 10, 11, 12].

The first section contains the definitions of spectrum and spectral subspaces associated with Banach space actions of compact groups, together with the basic theory and a few applications. In particular, we show that with our definitions of spectral subspace, the Tannaka duality theorem in [1], for properly infinite von Neumann algebras, if reformulated in our theory holds for a much larger class of algebras. In § 2 we specialise to compact connected Lie groups, and obtain the result analogous to the one in the commutative theory [6, 7, 8], that the spectrum of a strongly continuous representation of the real line on a Banach space is identical with the spectrum of its infinitesimal generator. In the final section we apply our theory to the action of compact groups by *-automorphisms on W^*-algebras. Recently, Størmer [15] has shown that it is possible to define the spectral subspaces corresponding to commutative actions from a natural unitary representation. Theorem 3.1 shows that this also can be done in the compact case. We also touch on the problem of defining an satisfying Γ-spectrum as in [4], and give some results in this direction.

§ 1 Spectral subspaces.

Throughout this paper, G will denote a compact group. A G-module is a finite dimensional complex vector space on which G acts continuously and linearly. One can then define direct sums and tensor products of G-modules, and consider submodules and isomorphisms between G-modules. Let $\hat{G}$ denote the space of isomorphism classes of all irreducible or simple G-modules. For any G-module, we choose an inner product $\langle .,. \rangle$.

In order to be able to handle strongly continuous automorphism groups on C^*-algebras, ultraweakly continuous automorphism groups on W^*-algebras, and even more general situations we follow the abstract setting of Arveson [2].

We consider a Banach space X , with X_* a linear subspace of the dual X^* of X , which may satisfy:

(1.1) X_* determines the norm on X i.e.

$$\|x\| = \sup\{|\langle \rho, x\rangle| \ : \ \rho \in X_* \ , \ \|\rho\| \leq 1\}, \ \forall x \in X \ .$$

(1.2) The $\sigma(X,X_*)$ closed convex hull of every $\sigma(X,X_*)$ compact set is $\sigma(X,X_*)$ compact.

We let $B(X)$ denote the bounded linear operators on X , and $B_\sigma(X)$ the $\sigma(X,X_*)$ continuous elements of $B(X)$.

Definition.

Let X be a Banach space, with associated space X_* , such that (X,X_*) satisfy (1.1) and (1.2). If α is a homomorphism from G into $B_\sigma(X)$ such that $g \to \langle \rho, \alpha(g)x \rangle$ is continuous on G , for all x in X, ρ in X_* , we say that

(α,X) is a representation of G .

Note that by compactness, (1.1) and the uniform boundedness theorem, that any representation (α,X) of G is uniformly bounded

i.e. $\sup\{\|\alpha(g)\| : g\in G\} < \infty$.

If (α,X) is a representation of G , we can lift α to $L^2(G)$ as follows: If $f\in L^1(G)$, there is an element denoted by $\alpha(f)$ in $B(X)$ such that

$$\langle\rho,\alpha(f)x\rangle = \int\langle\rho,\alpha(g)x\rangle f(g)dg$$

for all ρ in X_* , x in X , and where dg is the normalised Haar measure on G . If moreover (X,X_*) satisfy:

(1.3) The $\sigma(X_*,X)$-closed convex hull of every

$\sigma(X_*,X)$-compact set is $\sigma(X_*,X)$-compact.

then $\alpha(f)$ is $\sigma(X,X_*)$ continuous. [2, Proposition 1.4].

If (α,X) is a representation of G , then we define a G-module M in (α,X) to be a linear subspace of X , invariant under the action of α , and for which it is a G-module. If E is a set of G-modules, we let $M^\alpha(E)$ denote the $\sigma(X,X_*)$ closed linear span of all G-modules in (α,X) which are isomorphic to some member of E . If ι is the trivial irreducible G-module, then

$$M^\alpha(\iota) = \{x\in X : \alpha(g)x = x, \forall g\in G\} .$$

We then define the spectrum of the representation α , written $sp(\alpha)$, as

$$sp(\alpha) = \{\pi\in\hat{G} : M^\alpha(\pi) \neq 0\}.$$

Then $sp(\alpha) = \{\pi\in\hat{G} : \exists$ G-module in (α,X) isomorphic to $\pi\}$.

We shall see later in Proposition 1.5 and 1.6 that these spectral notions can also be formulated in terms of Fourier transforms, as in the abelian case [1, Def. 2.1.]. However the above definitions seem more natural and practical in the compact situation.

The following theorem shows the existence of G-modules under quite general circumstances, and that the simple G-modules determine the structure of the given Banach space representation. This generalises Shiga's result [13] for strongly continuous representations; with a little care, we can adapt his proof to our more general situation. As we shall see from Remark 1.11, it can also be regarded as an extension of a result of Roberts [12].

Theorem 1.2.

If (α,X) is a representation of G, we have $M^{\alpha}(\hat{G}) = X$.

Proof. If $\xi \in X$, we define a bilinear form in X_*, by $\langle f,g\rangle_\xi = \int \langle f,\alpha(b)\xi\rangle < g,\alpha(b)\xi\rangle db$. Let X_*^ξ be the subspace of X_*, consisting of all f in X_*, such that $\langle f,f\rangle_\xi = 0$. Let H_ξ be the hilbert space completion of X_*/X_*^ξ, and let θ_ξ be the canonical projection of X_* onto X_*/X_*^ξ. We have $\|\theta_\xi f\| \leq |\alpha|\,\|f\|$, for all f in X_*, where $|\alpha| = \sup\{\|\alpha(b)\| : b \in G\}$. Since $\alpha(a)^*$ maps X_*^ξ into X_*^ξ, we have a unitary representation U of G on H_ξ such that

$$U(a)^*\theta_\xi(f) = \theta_\xi[\alpha(a)^*f]$$

for all a in G, f in X_*. But

$$\langle U(a)^*\theta_\xi(f),\theta_\xi(g)\rangle = \int \langle f,\alpha(ba)\xi\rangle\overline{\langle g,\alpha(b)\xi\rangle}\,db ,$$

and $(a,b) \to \langle f,\alpha(ba)\xi\rangle\overline{\langle f,\alpha(b)\xi\rangle}$ is continuous on $G \times G$ for all f in X_*. It follows that U is strongly continuous on H_ξ. Decompose $H_\xi = \underset{\nu}{\oplus} H_{\xi,\nu}$ as an orthogonal direct sum of finite dimensional subspaces of H_ξ. Now by [2, Proposition 1.4] there is an element $x(\xi,f)$ in X such that

$$\langle g,x(\xi,f)\rangle = \int\langle g,\alpha(a)\xi\rangle\overline{\langle f,\alpha(a)\xi\rangle}\,da$$

for all f,g in X_*,ξ in X,

i.e. $\langle g,x(\xi,f)\rangle = \langle\theta_\xi(g),\theta_\xi(f)\rangle$.

This means $|\langle g,x(\xi,f)\rangle| \leq \|\theta_\xi(g)\|\;\|\theta_\xi(f)\| \leq |\alpha|\,\|\theta_\xi(f)\|\;\|g\|$, for all f,g in X_*, ξ in X. Then (1.1) implies that $\|x(\xi,f)\| \leq |\alpha|\,\|\theta_\xi(f)\|$, for all f in X_*, ξ in X. Thus since X is a Banach space, there is for all h in $H_\xi = [\theta_\xi(X_*)]^-$ a $F(\xi,h)$ in X satisfying $\langle g,F(\xi,h)\rangle = \langle h,\theta_\xi(g)\rangle$, for all g in X_*, ξ in X. Moreover

$$\|F(\xi,h)\| \leq |\alpha|\,\|h\| \qquad (1.4)$$

Now let Φ_ξ denote the injective mapping of H_ξ into X given by $\Psi_\xi(h) = F(\xi,h)$, so that $\|\Phi_\xi\| \leq |\alpha|$ by (1.4). The map Ψ_ξ takes the unitary representation U on H_ξ, to the representation α on X. In fact,

$$\begin{aligned}
\langle f,F(\xi,U(a)h)\rangle &= \langle U(a)h,\theta_\xi(f)\rangle \\
&= \langle h,U(a)^*\theta_\xi(f)\rangle \\
&= \langle h,\theta_\xi[\alpha(a)^*f]\rangle \\
&= \langle \alpha(a)^*f\;,\;F(\xi,h)\rangle \\
&= \langle f,\alpha(a)F(\xi,h)\rangle
\end{aligned}$$

for all h in H_ξ , f in X_* , and a in G .

It follows that $\Phi_\xi(H_{\xi,\nu})$ is an irreducible invariant finite dimensional subspace in X . Since Φ_ξ is continuous, the subspaces generated by $\Phi_\xi(H_{\xi,\nu})$ is norm dense in the vector space $\{F(\xi,h) : h \in H_\xi\}$. It only remains to show that the linear span of $\{F(\xi,h) : \xi \in X, h \in H_\xi\}$ is $\sigma(X,X_*)$ dense in X . If not, there exists a non-zero f in X_* with $\langle f,F(\xi,h)\rangle = 0$, for all ξ in X , h in H_ξ . In particular $\langle f,x(\xi,f)\rangle = 0$.

i.e. $\int |\langle f,\alpha(a)\xi\rangle|^2 da = 0$, for all ξ in X . By weak continuity of α , this can only happen if $f = 0$.

As promised, we now formulate our spectral notions in terms of fourier transforms. If $\pi \in \hat{G}$, let χ_π denote the associated "modified character" $\chi_\pi(g) = (\dim\pi)\,\mathrm{Tr}\,\pi(g)$, $g \in G$.

Lemma 1.3.

Let (α,X) be a representation of G , and (X,X_*) satisfy (1.3). Then $\alpha(\chi_\pi)$ is a projection of X onto $M^\alpha(\pi)$, with complementary subspace $M^\alpha(\hat{G}\setminus\{\pi\})$.

Proof. Let V be a simple G-module in (α,X) equivalent to π in $\hat{G}$. Then $\alpha(\chi_\pi)x = x$, for all x in V . Thus since $\alpha(\chi_\pi)$ is weakly continuous, $\alpha(\chi_\pi)x = x$ for all x in $M^\alpha(\pi)$. Moreover $M^\alpha(\pi)$ is a weakly closed subspace of X , invariant under α. Thus if V is a simple G-module in $\alpha(\chi_\pi)X$, we have $\alpha(\chi_{\pi_1})V = 0$, if $\pi_1 \not\cong \pi$, and $\alpha(\chi_\pi)x = x$ for all x in V . Thus V is equivalent to π , and hence by Theorem 1.2 applied to $\alpha(\chi_\pi)X$, we have $\alpha(\chi_\pi)X \subseteq M^\alpha(\pi)$. Thus $M^\alpha(\pi) = \alpha(\chi_\pi)X$. Similarly $M^\alpha(\hat{G}\setminus\{\pi\}) = [I - \alpha(\chi_\pi)]X$.

Corollary 1.4.

Let (α,X) be a representation of G, and (X,X_*) satisfy (1.3). Then $sp(\alpha)$ is finite if and only if α is norm continuous.

Proof. Suppose $sp(\alpha) = \{\pi_1,\dots,\pi_n\}$, i.e. $sp(\alpha)$ is finite. Let $f = \sum_{i=1}^{n} \chi_{\pi_i}$. Then $\alpha(f) = 1$, by Theorem 1.2 and Lemma 1.3. Thus $\alpha(g) - 1 = \alpha(f_g - f)$, for all g in G where $f_g(h) = f(g^{-1}h)$, $h \in G$. Then $\|\alpha(g)-1\| \leq |\alpha|\,\|f_g - f\|_1$, which tends to zero as g tends to 1; and where $|\alpha| = \sup\{\|\alpha(g)\| : g \in G\}$. Conversely, suppose α is norm continuous. Then if $\{f_\lambda\}$ is an approximate identity for $L^1(G)$, we have $\|\alpha(f_\lambda) - I\| \to 0$, and hence the Banach algebra A generated by $\alpha(L^1(G))$ has an identity. By [5, 3. 1. 8], $sp(A)$ is compact, and since $sp(A)$ is discrete, it is finite. Now $sp(\alpha) = sp(A)$, and the proof is complete.

The proof of the above Corollary is based on its abelian analogue in [7, 2. 4. 6].

Proposition 1.5.

Let (α,X) be a representation of G, and (X,X_*) satisfy (1.3). Then if π is an element of $\hat{G}$, the following conditions are equivalent:

(i) $\pi \in sp(\alpha)$; i.e. $M^\alpha(\pi) \neq 0$.

(ii) $\alpha(\chi_\pi) \neq 0$.

(iii) $\operatorname{Ker}\pi \supseteq \operatorname{Ker}\alpha$; i.e. $\pi(f) = 0$, whenever $\alpha(f) = 0$, $[f \in L^1(G)]$.

Proof. That (i) is equivalent to (ii) is a consequence of Lemma 1.3. Also, Theorem 1.2 and weak continuity show that $\operatorname{Ker} \alpha = \bigcap_{\pi \in \operatorname{sp} \alpha} \operatorname{Ker} \pi$. Thus (i) implies (ii). If $\pi \notin \operatorname{sp}(\alpha)$, then $\chi_\pi \in \operatorname{Ker} \alpha$ by (ii). But $\chi_\pi \notin \operatorname{Ker} \pi$, as $\pi(\chi_\pi) = 1$. Thus (iii) implies (i).

Propostition 1.6.

Let (α,X) be a representation of G , and (X,X_*) satisfying (1.3). Then for $E \subseteq \hat{G}$, and x in X , the following conditions are equivalent:

(i) $x \in M^\alpha(E)$

(ii) $\alpha(f)x = 0$, $\forall f \in \operatorname{Ker} E$, where $\operatorname{Ker} E = \bigcap_{\pi \in E} \operatorname{Ker} \pi$.

(iii) $x \in \{\alpha(f)y : y \in X , f \text{ s.t. } \operatorname{supp} f \subseteq E\}^-$ (the $\sigma(X,X_*)$ closure).

Proof. That (i) implies (iii) is clear from Lemma 1.3 and that (iii) implies (ii) follows by weak continuity of $\alpha(f)$. Let Y denote the set of x in X which satisfy (ii). Then Y is a $\sigma(X,X_*)$ closed linear subspace in X , which is invariant under the action of G . It is then seen by considering characters that any simple G-module in $(\alpha|_Y,Y)$ must be in E . Thus (ii) implies (i) follows from Theorem 1.2 applied to $(\alpha|_Y,Y)$.

Corollary 1.7.

Let (α,X) be a representation of G , and (X,X_*) satisfying (1.3). Then we have:

(i) If $E \subseteq \hat{G}$, the spectral subspace $M^\alpha(E)$ is the maximal $\sigma(X,X_*)$ -closed, G-invariant subspace Y of X such that $\alpha|_Y$ contains only irreducibles of E .

(ii) If $E_1, E_2 \subseteq \hat{G}$, then $M^\alpha(E_1) \cap M^\alpha(E_2) = M^\alpha(E_1 \cap E_2)$.

Proof.

(i) That $\alpha|_{M^\alpha(E)}$ contains only irreducibles of E follows from (ii) of Proposition 1.6. Suppose Y in X is $\sigma(X,X_*)$ closed and contains only irreducibles of E . Then every simple G-module in $(\alpha|_Y, Y)$ is in $M^\alpha(E)$ by the very definition of a spectral subspace. Thus Theorem 1.2 applied to $(\alpha|_Y, Y)$ shows that $Y \subseteq M^\alpha(E)$.

(ii) This is a consequence of (ii) of Proposition 1.6.

Let (α, X) be a representation of G , and x an element in X . We can define the spectrum of x written $sp_\alpha(x)$ as follows:

$$sp_\alpha(x) = \{\pi \in \hat{G} : \alpha(\chi_\pi)x \neq 0\} \qquad (1.5)$$

Then if (X, X_*) satisfies (1.3) one can show using Theorem 1.4 that:

$$M^\alpha(E) = \{x \in X : sp_\alpha(x) \subseteq E\}$$

for any subset E of $\hat{G}$. By considering characters, it is seen that

$$\{\pi \in \hat{G} : \pi(f) = 0 \text{ , if } \alpha(f)x = 0 \text{ , } f \in L^1(G)\} \quad (1.6)$$

is always a subset of $sp_\alpha(x)$. For compact abelian groups, these concepts coincide, i.e. the spectrum of an element as defined by (1.5) is the same as that considered by Arveson in [2]. However in general, $sp_\alpha(x)$ is not equal to (1.6), which is indeed often empty. [The reader should consider the group $G = S_3$, the symmetric group on three letters, with generators x,y, and $\alpha = \pi$, the two dimensional irreducible representation of G as in Example 3.5. Then consider $f = \omega\,\epsilon(x) + \epsilon(y)$, where $\epsilon(x)$ and $\epsilon(y)$ are the Dirac delta functions at x and y respectively, and $\omega = e^{2\pi i/3}$) .]

For later use in an application of our theory to von Neumann algebras, (Theorem 3.6), we record at this point the following Proposition. Here σ denotes the spectrum of an operator on a Banach space. This should be compared with the analogous result in the commutative situation [4,2. 3. 8].

<u>Propostion 1.8</u>.

Let (α,X) be a representation of G such that (1.3) holds. Then if $g \in G$, we have

$$\sigma[\alpha(g)] = (\cup\{\sigma[\pi(g)] : \pi \in sp(\alpha)\})^{-} .$$

<u>Proof</u>. Suppose $\lambda \in \sigma[\pi(g)]$ for some π in $sp(\alpha)$. Then there exists a non-zero x in a simple G-module in (α,X), equivalent to π, such that $\alpha(g)x = \lambda x$. Thus $\lambda \in \sigma[\alpha(g)]$, and hence $(\cup\{\sigma[\pi(g)] : \pi \in sp\alpha\})^{-} \subseteq \sigma[\alpha(g)]$. For the reverse inclusion we follow Connes' argument [4]. Suppose $\lambda \notin (\cup\{\sigma[\pi(g)] : \pi \in sp(\alpha)\})^{-}$, and let W be an open set in $T = \{z \in C : |z| = 1\}$, containing $(\cup\{\sigma[\pi(g)] : \pi \in sp(\alpha)\})^{-}$, but $\lambda \notin W$. Let f be a C^{∞}-function on T which coincides with $(z-\lambda)^{-1}$ on W. Then $f(z) = \Sigma a_n z^n$ for $z \in W$, with $\Sigma|a_n| < \infty$. If $x = \Sigma a_n\,\alpha(g^n)$, then $x \in B(X)$, and in fact $x = \alpha[\Sigma a_n\,\epsilon(g^n)]$, where for t in G, $\epsilon(t)$ is the Dirac delta function at t, and where for any finite measure μ on G, $\alpha(\mu)$ is the weak integral defined in [2]. Then

$$\alpha(\mu_1) = x[\alpha(g)-\lambda] \text{ , and } \alpha(\mu_2) = [\alpha(g)-\lambda]x \text{ ,}$$

where $\mu_1 = [\Sigma a_n \, \epsilon(g^n)]*[\epsilon(g)-\lambda]$,

and $\mu_2 = [\epsilon(g)-\lambda]*[\Sigma a_n \, \epsilon(g^n)]$.

In particular, $\gamma(\mu_1) = f[\gamma(g)][\gamma(g)-\lambda]$, for all γ in $\hat{G}$, an hence $\pi(\mu_1) = 1$ if $\pi \in sp(\alpha)$. Thus $\alpha(\mu_1) = 1$, by Theorem 1.2, and since $\alpha(\mu_1)$ is weakly continuous by [2]. Similarly $\alpha(\mu_2) = 1$, and hence $\lambda \notin \sigma[\alpha(g)]$.

If M is a G-module, we let $C(M)$ denote the linear span of all complex valued functions on G of the form $g \to \langle gm, n\rangle$; $m,n \in M, g \in G$.

<u>Lemma 1.9</u>.

Let G act by automorphisms on an algebra X , and let M_1 and M_2 denote two subspaces of X which are G-modules. Then M_1M_2 is a G-submodule of $M_1 \otimes M_2$, and $C(M_1)C(M_2) \subseteq C(M_1M_2)$.

<u>Proof</u>. Define an inner product on M_1M_2 by

$$\langle \sum_i x_i y_i , \sum_j x'_j y'_j \rangle = \sum_{i,j} \langle x_i, x'_j\rangle \langle y_i, y'_j\rangle \quad (1.7)$$

This shows that G acts continuously on M_1M_2 . Define $\Phi : M_1 \otimes M_2 \to M_1M_2$, by $x \otimes y \to xy$. We choose an injection $\eta = M_1M_2 \to M_1 \otimes M_2$, and integrate over the compact group G to obtain an intertwining operator:

$$\bar{\eta}(m) = \int_G g.\eta(g^{-1}m)dg \text{ .}$$

Indeed,

$$h.\bar{\eta}(m) = \int_G (hg).\eta(g^{-1}m)dg = \int g.\eta(g^{-1}hm)dg = \bar{\eta}(hm)$$

for all h in G , m in M_1M_2 .

If we choose η such that $\Phi\circ\eta = 1$, it is easily seen that $\Phi\circ\bar{\eta} = 1$, so that $\bar{\eta}$ is injective. This shows that M_1M_2 is a submodule of $M_1\otimes M_2$. Finally $C(M_1)C(M_2) \subseteq C(M_1M_2)$ follows from (1.7).

If E_1 and E_2 are two subsets of $\hat{G}$, we let $E_1 + E_2$ denote $\cup\{sp(\pi_1\otimes\pi_2) : \pi_1 \in E_1 , \pi_2 \in E_2\}$. The following is then an immediate consequence of Lemma 1.9.

Proposition 1.10.

Let (α,X) be a representation of G , and suppose moreover that X is an algebra where multiplication is separately weakly continuous and G acts by automorphisms. Then if $E_1,E_2 \subseteq \hat{G}$, we have:

$$M^{\alpha}(E_1)M^{\alpha}(E_2) \subseteq M^{\alpha}(E_1 + E_2) .$$

Remark 1.11.

Lemma 1.9 and Proposition 1.10 could be set in more general circumstances. Suppose (β,X) and (α,Y) are representations of G , and that (X,X_*) moreover satisfies (1.3) and both X_* and Y_* are Banach spaces. Suppose also that either β is strongly continuous on X , or α^* is strongly continuous on Y_* . We let Z denote $B_\sigma(X,Y)$ the weakly continuous operators from X into Y , with Z_* as the projective tensor product $X\overset{\vee}{\otimes} Y_*$ which can be isometrically identified with a closed subspace of Z^* . Let A be a $\sigma(Z,Z_*)$ closed linear subspace of Z , such that $\gamma(g)$ defined by

$$\gamma(g)(z) = \alpha(g)z\beta(g)^{-1} , z\in Z$$

leaves $\mathbf{A}$ invariant. Then $(\gamma, \mathbf{A})$ is a representation of G [2, Proposition 1.6].

If $E_1, E_2 \subseteq \hat{G}$, we have as in Lemma 1.9 and Proposition 1.10 that

$$M^{\gamma}(E_1)M^{\beta}(E_2) \subseteq M^{\alpha}(E_1 + E_2) . \qquad (1.8)$$

(Proposition 1.9 can in fact be deduced from (1.8) under suitable conditions, by considering the left regular representation as in the abelian case (e.g. [2, Lemma 2])).

We use this product rule to show the connection of our work with some spectral subspaces of Roberts [10, 12]. Suppose that M is a von Neumann algebra, and (α, M) a representation of G by automorphisms, so that M_* denotes the predual of M as usual. Let β be a representation of G on a finite dimensional Hilbert space H_β. In the previous notation, we have $X = M$, $Y = H_\beta$. We identify $M \otimes \bar{H}_\beta$ with $B(H_\beta, M)$ in the usual way, and under this identification the representation γ of G on $M \otimes H_\beta$ corresponds to $\alpha \otimes \bar{\beta}$. Taking $E_1 = \{\iota\}$, and $E_2 = \{\mathrm{sp}\ \beta\}$, we thus have

$$M^{\gamma}(\iota)M^{\beta}(\mathrm{sp}\ \beta) \subseteq M^{\alpha}(\mathrm{sp}\ \beta) .$$

i.e. if we let $M_\beta = M^{\gamma}(\iota)$ in the notation of Roberts, we have $M_\beta H_\beta \subseteq M^{\alpha}(\mathrm{sp}\ \beta)$.

On the other hand, suppose V is a G-module in $M^{\alpha}(\mathrm{sp}\ \beta)$ equivalent with β . Then there exists k in $B(H_\beta, V)$ such that $k\beta(g) = \beta(g)k$ for all g in G . i.e. $k \in M^{\gamma}(\mathrm{sp}\ \beta)$. Then $V = kH_\beta \subseteq M^{\gamma}(\mathrm{sp}\ \beta) . H_\beta$. Hence $M^{\alpha}(\mathrm{sp}\ \beta) = [M_\beta H_\beta]^{-}$ the ultraweak closure.

We recall the natural operation of conjugating G-modules.

Now suppose (α,X) is a representation of G, and that moreover X is a $*$-vector space, with G acting on X by *-maps; i.e. $\alpha(g)(x)^* = [\alpha(g)(x)]^*$, for all x in X, g in G. Then if M is a G-module in (α,X) we can define an inner product on M^* as follows:

$$\langle x,y\rangle = \overline{\langle x^*,y^*\rangle} \quad , \quad x,y\in M^* \qquad (1.9)$$

so that M^* is also a G-module in (α,X) and is conjugate to M, with $C(M^*) = \overline{C(M)}$. Thus $sp(\alpha)$ is symmetric, i.e. $sp(\alpha) = \overline{sp(\alpha)}$, and moreover $M^\alpha(E)^* = M^\alpha(\bar{E})$ if the *-operation is weakly continuous.

The following generalises the Tannaka duality theorem in [1, Appendix C]:

<u>Theorem 1.12.</u>

Let (α,X) be a representation of G. Suppose moreover that X is an algebra with involution, and G acts on X by *-automorphisms. Let θ be a weakly continuous *-automorphism of X such that

(i) $\theta(M)\subseteq M$, for all G-modules M in (α,X).

(ii) $\theta(x) = x$, for all x in $M^\alpha(\iota)$.

Then there exists g_θ in G such that $\theta = \alpha(g_\theta)$.

<u>Proof</u>. We define $C_\alpha(G)$ to be the closed linear span of all $C(M)$, where M runs over all G-modules in $sp(\alpha)$. We claim that $C_\alpha(G)$ is a C^*-subalgebra of $C(G)$, the C^*-algebra of all complex valued functions on G. That $C_\alpha(G)$ is a *-vector subspace follows from the remarks before the theorem, and that $C_\alpha(G)$ is a subalgebra follows from Lemma 1.9.

We now define a map U on $C_\alpha(G)$ in the following way. If M is a G-module in (α,X), and $m,n\in M$, then U will take $\langle\alpha(g)m,n\rangle$ to $\langle\alpha(g)[\theta(m)],n\rangle$. First we note that if M is any G-module in (α,X) then

$$\int\langle\alpha(g)[\theta(m)],n\rangle dg = \int\langle\alpha(g)m,n\rangle dg \qquad (1.10)$$

for all m,n in M. This can be seen by decomposing M into irreducible submodules, so that we can assume that M is simple. Then take $m,n\in M$. Then $\alpha(\chi_M)(n) = 0$ unless $M = \iota$, in which case $\theta(m) = m$. This means $\langle\theta(m),\alpha(\chi_M)n\rangle = \langle m,\alpha(\chi_M)n\rangle$, for all simple G-modules M in (α,X), and all m,n in M, which is equivalent to (1.10).

Let M_1,M_2 be two G-modules in (α,X), and let $x_1,y_1\in M_1$, $x_2,y_2\in M_2$, We have:

$$\int\langle\alpha(g)[\theta(x_1)],y_1\rangle\overline{\langle\alpha(g)[\theta(x_2)],y_2\rangle}dg$$

$$= \int\langle\alpha(g)[\theta(x_1)],y_1\rangle\langle\alpha(g)[\theta(x_2^*)],y_2^*\rangle dg \quad \text{, by (1.9), and since } \theta \text{ is a }^*\text{-map.}$$

$$= \int\langle\alpha(g)[\theta(x_1x_2^*)],y_1y_2^*\rangle dg \qquad \text{by (1.7)}$$

$$= \int\langle\alpha(g)[x_1x_2^*],y_1y_2^*\rangle dg \qquad \text{by (1.10)}$$

$$= \int\langle\alpha(g)(x_1),y_1\rangle\overline{\langle\alpha(g)x_2,y_2\rangle}dg$$

This means that U is well defined and is in fact an isometry on $C_\alpha(G)$ for the L^2-norm. It is easy to check U is a *-homomorphism, and commutes with right translations. It follows from Lemma A of [1, Appendix A] that there exists g_θ in G such that

$$\langle\theta(x),y\rangle = \langle\alpha(g_\theta)(x),y\rangle$$

for all x,y in a G-module in (α,X). The result follows from

Theorem 1.2, and weak continuity of θ and $\alpha(g_\theta)$.

Remark 1.13.

The duality theorem of [1] can be deduced from the above. In fact suppose X is a von Neumann algebra, and (α, X) a representation of G by *-automorphisms. Suppose that θ is a *-automorphism of X such that

(i) $\theta(H) \subseteq H$, for each finite dimensional Hilbert space H in X (as defined in [11]) which is globally invariant under α.

(ii) $sp(\alpha) = Msp(\alpha)$, the monoidal spectrum of α; i.e. the irreducible equivalence classes of restrictions of α to Hilbert spaces in X.

(iii) $\theta(x) = x$, for all x in $M^\alpha(\iota)$.

Then there exists g_θ in G such that $\theta = \alpha(g_\theta)$. In fact by the above theorem, it is enough to show that θ leaves invariant each simple G-module M in (α, X). But the argument in [1, Appendix C] shows that there exists a in $M^\alpha(\iota)$, and a Hilbert space H in X, invariant under α, such that $V = aH$. Since $\theta(a) = a$, it is clear that θ leaves V globally invariant.

§ 2. Application to compact Lie groups.

Let G be a compact connected Lie group, with Lie algebra $\mathcal{G}$. Let X be a Banach space, and α a strongly continuous representation of G on X. Then we can lift α to a representation $\partial\alpha$ of the Lie algebra $\mathcal{G}$, by unbounded operators as follows: If $A \in \mathcal{G}$, we let $\partial\alpha(A)$ be the infinitesimal generator of the strongly continuous one parameter group $t \to \alpha[\exp(tA)]$. Thus

$$\alpha[\exp(tA)] = \exp[t\partial\alpha(A)] \quad \text{for } A \text{ in } \mathcal{G},\ t \text{ in } \mathbb{R}.$$

We put $\mathcal{D}(\partial\alpha) = \cap\{\mathcal{D}\partial\alpha(A) : A \in \mathcal{G}\}$. Let θ be a representation of the Lie algebra $\mathcal{G}$ on a finite dimensional vector space V. Then we say θ ocurs in $\partial\alpha$, written $\theta \subseteq \partial\alpha$, if $V \subseteq \mathcal{D}(\partial\alpha)$, $\partial\alpha$ leaves V invariant, and $\theta \approx \partial\alpha|V$.

Lemma 2.1

Let θ be a finite dimensional representation of $\mathcal{G}$. Then $\theta \subseteq \partial\alpha$ if and only if there is a G-module π in (α, X) such that $\partial\pi = \theta$.

Proof. Let $\theta \subseteq \partial\alpha$. We must show that α leaves V, the representation space of θ, globally invariant. This follows from

$$\alpha[\exp(tA)] = \operatorname*{st\,lim}_{n\to\infty}\left[I - \frac{t\partial\alpha(A)}{n}\right]^{-n}$$

valid for all A in $\mathcal{G}$, t in $\mathbb{R}$, which shows that α leaves V invariant, and $\alpha[\exp tA]|_V = \exp t\theta(A)$. Thus $\partial(\alpha|_V) = \theta$. The converse is clear. $\square$

If $\hat{\mathcal{G}}$ is the space of isomorphism classes of irreducible finite dimensional representations of the Lie algebra $\mathcal{G}$, we define the spectrum of $\partial\alpha$, written $sp(\partial\alpha)$ as

$$sp(\partial\alpha) = \{\theta \in \hat{\mathcal{G}} : \theta \subseteq \partial\alpha\} .$$

If $\partial sp(\alpha) = \{\partial\pi : \pi \in \hat{G}\}$, the above lemma allows us to deduce:

Theorem 2.2.

$$sp(\partial\alpha) = \partial sp(\alpha)$$

In [6] (see also [7, 8] for the norm continuous case), it was shown that the spectrum of a strongly continuous representation of the real line is equal to the spectrum of its infinitesimal generator. The identity $sp(\partial\alpha) = \partial sp(\alpha)$ is the analogue of this for compact Lie groups. Let us look at this more closely for the circle group $G = \mathbb{T}$, with Lie algebra $\mathcal{G} \approx \mathbb{R}$. Then if α is a strongly continuous representation of $\mathbb{T}$ on a Banach space $\mathbf{X}$, there is a closed, densely defined operator Z on X such that

$$\alpha(e^{i\theta}) = e^{i\theta Z} \text{ for all } \theta \text{ in } \mathbb{R} .$$

Then Theorem 2.2 means that $sp(\alpha) = P\sigma(Z)$, the point spectrum of Z .

§ 3. Applications to von Neumann algebras.

In this section, we consider applications of our theory to the action of compact groups by *-automorphisms on von Neumann algebras.

Størmer [15] has shown that not only can spectral subspaces of the action of locally compact abelian groups by *-automorphisms on von Neumann algebras be regarded as a generalization of those for unitary representations, but that they can also be defined from the spectral subspaces of a particular unitary representation. We begin by showing that the same situation holds in the compact case.

Let A be a von Neumann algebra, on a Hilbert space H, and ω a strongly continuous unitary representation of G on H such that $\omega_g^* A \omega_g = A$ for all g in G. Let $\beta_g(x) = \omega_g^* x \omega_g$, $x \in B(H)$, and $\alpha = \beta|_A$. (Note that any weakly continuous representation α of G by *-automorphisms on A could be put in this form by a crossed product construction). β induces, by restriction, a unitary representation U of G on H_2, the Hilbert-Schmidt operators on H. U is strongly continuous, since $U_g = \omega_g^* \otimes \bar{\omega}_g^*$, if we identify H_2 with the Hilbert space tensor product $H \otimes \bar{H}$.

Theorem 3.1.

If E is a subset of $\hat{G}$, $M^\alpha(E) = A \cap M^U(E)^-$, where the closure is in the ultraweak topology on $B(H)$.

Proof. Let $y \in M^U(E)$. Then by Proposition 1.6 (ii),

$U(f)y = 0$, $\forall f \in \operatorname{Ker} E$; hence $\beta(f)x = 0$,

$\forall x \in M^U(E)^-$, $\forall f \in \operatorname{Ker} E$. Hence $\alpha(f)x = 0$,

$\forall x \in A \cap M^U(E)^-$, $\forall f \in \operatorname{Ker} E$.

Thus by Prop. 1.6. again we have $A \cap M^U(E)^- \subseteq M^\alpha(E)$. Conversely, let $\pi \in E$, $x \in M^\alpha(\pi)$. Then $\alpha(\chi_\pi)x = x$. Let $x_\nu \in H_2$, $x_\nu \underset{\nu}{\rightarrow} x$ ultraweakly. Then $U(\chi_\pi)x_\nu \in M^U(\pi)$, and $U(\chi_\pi)x_\nu \underset{\nu}{\rightarrow} x$. Hence $x \in A \cap M^U(\pi)^- \subseteq A \cap M^U(E)^-$. That is $M^\alpha(\pi) \subseteq A \cap M^U(E)^-$, and we have proved $M^\alpha(E) \subseteq A \cap M^U(E)^-$. $\square$

Let β be a representation of a compact group G by *-automorphisms on a C^*-algebra A, with φ any G-invariant state. If $(\pi_\varphi, \xi_\varphi)$ is the GNS decomposition for φ , then the following proposition could be applicable to the von Neumann algebra $M = \pi_\varphi(A)''$, with cyclic vector ξ_φ , and unitary representation given by $U_g : \pi_\varphi(x)\xi_\varphi \rightarrow \pi_\varphi(\beta_g(x)\xi_\varphi)$, $x \in A$, $g \in G$. This was also the construction of the unitary group in Theorem 3.1, where we employed the GNS decomposition for the invariant weight, namely the trace, on $B(H)$. The first part of the following result was proved by Størmer [14] in the abelian case.

Proposition. 3.2.

Let G be a compact group, M a von Neumann algebra with cyclic vector x_o . Assume $\alpha : G \rightarrow \operatorname{Aut} M$ is an ergodic representation of G which is implemented by a strongly continuous unitary representation $\alpha_g(A) = U_g A U_g^{-1}$, $g \in G$, $A \in M$; and suppose $U_g x_o = x_o$, $\forall g \in G$.

(1) Let $\pi \in \operatorname{sp} \alpha$ be realized on the irreducible subspace $M_\pi \subseteq M$. Then $M_\pi x_o$ is an irreducible U-invariant subspace of H , and π is unitarily equivalent to the subrepresentation of U on $M_\pi x_o$.

(2) The injective map $\pi \to U|_{M_\pi x_o}$; $sp\alpha \to sp\, U$ is a bijection if and only if $\mathrm{Ker}\,\alpha = \mathrm{Ker}\, U$ in $L^1(G)$.

Proof. Let $\pi \in sp\,\alpha$, $0 \neq A \in M_\pi$; hence $\alpha_g(A) = \pi(g)A$, $\forall g \in G$. We define an inner product $\langle .,. \rangle$ on M_π as follows. Put, for $x,y \in G$,

$$\langle \alpha_x(A),\alpha_y(A) \rangle = \langle \pi(x)A,\pi(y)A \rangle \overset{\mathrm{def}}{=} (U_xAx_o,U_yAx_o) ,$$

where $(.,.)$ is the inner product of H . Then

$$\langle \pi(x)A,\pi(x)A \rangle = \|U_xAx_o\|^2 = \|Ax_o\|^2 \neq 0$$

since x_o is separating for M . (α acts ergodically). Also, $\langle \pi(x)A,A \rangle = (U_xAx_o,Ax_o)$, so that the (cyclic) π is unitarily equivalent to $U|_{M_\pi x_o}$ which is in $sp\, U$. ((1) can also be deduced from the proof of Lemma 1.9, letting $M_1 = M_\pi$, $M_2 = Cx_o$, hence $M_1 \otimes M_2 \simeq M_1$.)

(2): We may lift α to a faithful representation $\tilde{\alpha}$ of $L^1(G)/\mathrm{Ker}\,\alpha$. If $\mathrm{Ker}\,\alpha = \mathrm{Ker}\,U$ we have $sp\,\alpha = sp\,\tilde{\alpha} = (L^1(G)/\mathrm{Ker}\,\alpha)^\wedge = sp\,\tilde{U} = sp\,U$. Conversely, if $sp\,\alpha = sp\,U$ we see $(L^1(G)/\mathrm{Ker}\,\alpha)^\wedge = (L^1(G)/\mathrm{Ker}\,U)^\wedge$, and since $\mathrm{Ker}\,U \subseteq \mathrm{Ker}\,\alpha$, $L^1(G)/\mathrm{Ker}\,\alpha = L^1(G)/\mathrm{Ker}\,U$, hence $\mathrm{Ker}\,\alpha = \mathrm{Ker}\,U$. $\square$

Connes [4] has introduced the useful notion of a Γ-spectrum for locally compact, abelian actions. This has been particularly useful in the problem of unitary implementation of *-automorphisms, and relating properties of the algebra with those of the fixed point algebra. Here we look at a Γ-spectrum for compact actions.

Let A be a von Neumann algebra on a Hilbert space H , and let (α,A) be a representation of G by *-automorphisms. We let A^α denote the fixed point algebra, namely $M^\alpha(\iota)$.

Following Connes [4,2.2.1] we define the Γ-spectrum as follows:

$$\Gamma(\alpha) = \bigcap_e \mathrm{sp}(\alpha_e) .$$

where the intersection is over all non-zero projections e in A^α, and α_e denotes the restriction of α to the invariant reduced algebra $A_e = eAe$. Note that by Propostion 1.10,

$$M^\alpha(E) \cap A_e = eM^\alpha(E)e = M^{\alpha e}(E)$$

for all subsets E og $\hat{G}$, and projections e in A^α. Thus

$$\Gamma(\alpha) = \{\pi \in \hat{G} : eM^\alpha(\pi)e \neq 0 , \forall \text{ non-zero projections } e \text{ in } A^\alpha\}$$

If e_1, e_2 are two projections in A^α, which are equivalent in A, then in general $\Gamma(\alpha_{e_1}) \neq \Gamma(\alpha_{e_2})$ (see Example 3.5). This should be contrasted with [4, Lemma 2.2.5] for the abelian situation. However we can show the following.

<u>Proposition 3.3.</u>

Let e_1, e_2 be projections in A^α which are equivalent in A^α. Then $\mathrm{sp}(\alpha_{e_2})$ and $\Gamma(\alpha_{e_1}) = \Gamma(\alpha_{e_2})$.

<u>Proof.</u> Let u be a partial isometry in A^α such that $e_2 = uu^*$, and $e_1 = u^*u$. Then

$$u\{\text{non-zero projections } e \text{ in } A^\alpha : e \leq e_1\}u^* = \{\text{non-zero projections } e \text{ in } A^\alpha : e \leq e_2\} .$$

It is thus enough to show that $\mathrm{sp}(\alpha_{e_1}) = \mathrm{sp}(\alpha_{e_2})$. This is clear since if V is a G-module in (α_{e_1}, A_{e_1}) equivalent with π in $\hat{G}$, then by Lemma 1.9, u^*Vu is a G-submodule of π . Hence by irreducibility, u^*Vu must be equivalent to π . Thus $sp(\alpha_{e_1}) \subseteq sp(\alpha_{e_2})$. Hence $sp(\alpha_{e_1}) = sp(\alpha_{e_2})$ by symmetry.

Proposition 3.4.

(a) If e is a non-zero projection in A^α , and $\bar{e}$ its central cover in A^α , then $sp(\alpha_e) = sp(\alpha_{\bar{e}})$.

(b) $\Gamma(\alpha) = \bigcap_e sp(\alpha_e)$, where e ranges over all non-zero central projections in A^α .

(c) If A^α is a factor, then $\Gamma(\alpha) = sp(\alpha)$.

Proof.

(a): One has $\bar{e} = \mathbf{v}ueu^*$, $u \in A^\alpha$, unitary. Let $x \in M^\alpha(\pi) \cap A_{\bar{e}}$, $\pi \in sp(\alpha_{\bar{e}})$. There are unitaries u and $\mathbf{v}$ in A^α such that $ueu^*x\mathbf{v}e\mathbf{v}^* \neq 0$, $u^*x\mathbf{v} \neq 0$. By $u^*x\mathbf{v} \in M^\alpha(\pi)$; also $0 \neq eu^*x\mathbf{v}\,e \in Ae$, and $eu^*x\mathbf{v}\,e \in eM^\alpha(\pi)e \subseteq M^\alpha(\pi)$. Hence $eu^*x\mathbf{v}\,e\mathbf{v}^* \in M^\alpha(\pi) \cap A_e = M^{\alpha e}(\pi)$, and $sp(\alpha_e) = sp(\alpha_{\bar{e}})$.

(b) and (c) are obvious from (a).

We now give an example which shows that the Γ-spectrum is not an invariant under outer equivalence of representations. This is contrary to what happens in the abelian case. The difference is closely related to the fact that non-abelian compact groups may permit ergodic representation on von Neumann algebras by inner automorphisms.

Example 3.5.

Let $G = S_3$, the symmetric group on three letters. If $x,y \in G$, $x^2 = e$, $y^3 = e$, $xy = y^2x$, we can realise $\hat{G}$ as follows. $\hat{G} = \{\iota, \lambda, \pi\}$, where ι is the identity representation, $\lambda(x) = -1$, $\lambda(y) = 1$, and π is the two-dimensional representation

$$\pi(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \pi(y) = \begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix} ; \quad \omega = e^{2\pi i/3} .$$

Let $A = B(C^2)$, π implements a representation $(\alpha, B(C^2))$ of G by *-automorphisms:

$$\alpha(g)(a) = \pi(g) a \, \pi(g)^{-1} \quad , a \in A , \; g \in G .$$

One has $\alpha \simeq \pi \otimes \bar{\pi}$, and by computing $\int_G \chi_{\pi \otimes \bar{\pi}} \chi_\rho$ for all ρ in $\hat{G}$ one sees that $sp(\alpha) = \hat{G}$.
Furthermore α acts ergodically since π is irreducible, hence $A^\alpha = C1$. Thus $\Gamma(\alpha) = sp(\alpha) = \hat{G}$, by Proposition 3.4. However $\Gamma(\alpha) \neq \Gamma(1) = \{\iota\}$, but still α is outer equivalent to 1 .

One half of the proof in [4, Thm. 2.3.1] for the abelian situation can be adapted, with the help of Proposition 1.8, to show the following theorem. However, we have no result regarding the truth of the reverse inclusion in (3.1).

Theorem 3.6.

Let M be a factor von Neumann algebra. Suppose $g \in G$, and u is a unitary in A^α , such that $\alpha(g) = \mathrm{ad}\, u$. Then $g \in \Gamma(\alpha)^\circ$, the annihilator of $\Gamma(\alpha)$ in G . Hence

(3.1) $\{g \in G : \alpha(g) = \mathrm{ad}\, u$, for some unitary in $A^\alpha\} \subseteq \Gamma(\alpha)^\circ$

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